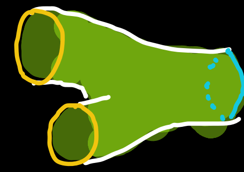


MU-Theory and Formal Group Laws

Following Chpt. 3 of Orange Book

Scotty Tilton
UCSD

$\sum_{i,j\geq 0} a_{ij}x^i y^j$

$L \xrightarrow{\exists!\,\theta} R$

$\mathbb{CP}^\infty$

Definition

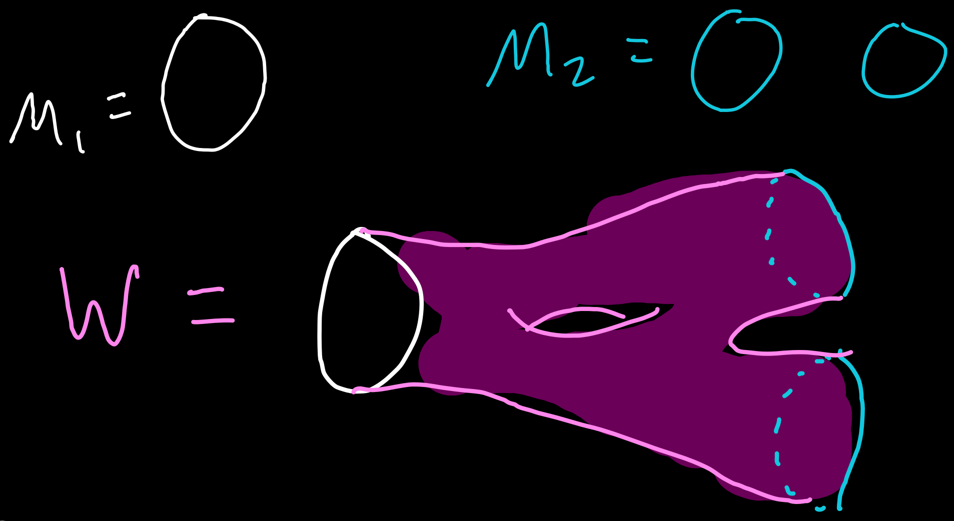
Let M_1, M_2 be smooth n-manifolds
and $f_1 : M_1 \longrightarrow X$, $f_2 : M_2 \longrightarrow X$ be continuous maps

These maps are <u>bordant</u> if

- there exists a smooth, compact manifold W with $\partial W = M_1 \sqcup M_2$
- and $f : W \longrightarrow X$ such that $f|_{M_i} = f_i$ for $i = 1, 2$.

We call f a <u>bordism</u> between f_1 f_2.

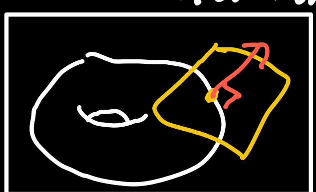
Picture



Fact Bordism is an equivalence relation and the set of bordism classes form a group under $\sqcup$.

$$\boxed{\begin{aligned} [M_1] + [M_2] &= [M_1 \sqcup M_2] \\ [M_1] + [M_1] &= [\partial (M_1 \times I)] = 0 = [\emptyset] \end{aligned}}$$

Note: A manifold is <u>stably complex</u> if it admits a complex linear structure on its stable normal bundle (weaker than complex analytic)



Definition $MU_n(X)$, the <u>nth complex bordism group of X</u> is the bordism group where you require all manifolds to be stably complex.

Remarks
- $MU_*(-)$ satisfy the E-S axioms (except dimension)
- We denote $MU_* := MU_*(pt)$ the group of bordism classes of stably complex manifolds.
- Graded under "$\times$"

Theorem 3.1.3 $MU_* \cong \mathbb{Z}[x_1, x_2, \dots]$ where $\dim x_i = 2i$

Remarks
- $\mathbb{C}P^i$ form polynomial generators for $\mathbb{Q} \otimes MU_*$
- $MU_*(X)$ is an MU_* module.

$\lambda \in MU_*$ rep'd manifold N

$x \in MU_*(X)$ rep'd $f : M \to X$

λx is rep'd by
$$\begin{tikzcd} N \times M \arrow[r] \arrow[rr, bend right] & M \arrow[r, "f"] & X \end{tikzcd}$$

We have MU-theory now

Definition A formal group law over a commutative ring with identity R is a – power series $F(x,y)$ over R satisfying
- (Identity) $F(x,0) = F(0,x) = x$
- (Commutativity) $F(x,y) = F(y,x)$
- (Associativity) $F(F(x,y),z) = F(x, F(y,z))$

Remark
- existence of inverses is free with these
  $i(x)$ is determined $F(x, i(x)) = 0$
- related to a formal group law F is a logarithm $\log_F$ so that

$$\boxed{\log_F(F(x,y)) = \log_F(x) + \log_F(y)}$$

⚠ not in general

Examples

- Additive formal group law
  $F(x,y) = x+y$ $\quad \log_F(x) = x$

- Multiplicative formal group law
  $F(x,y) = x+y+xy$ $\quad \log_F(x) = \log(1+x)$
  $= (1+x)(1+y) - 1$

- For more: take the power series of the produce map at the identity of a Lie group or abelian variety

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Lazard's Theorem

(Universal Formal Group Law) There is a formal group law $G(x,y) = \sum_{i,j} a_{ij} x^i y^j$ over a ring L called the Lazard ring such that for any formal group law $F(x,y)$ over a ring R, then there exists a unique ring homomorphism $\theta: L \longrightarrow R$ such that

$$F(x,y) = \sum_{i,j} \theta(a_{ij}) x^i y^j$$

- L is a polynomial algebra $\mathbb{Z}[x_1, x_2, \ldots\ldots]$ where if we put a grading on L so that $\deg(a_{ij}) = 2-2i-2j$ makes $\deg(x_i) = -2i$

How are these related?

Def Associated to MU_* is a cohomology theory MU^*

Geometrically X an m-manifold.

$[f] \in MU^k(X)$ is represented by a function $f: N \to X$ where N is an $m-k$-manifold that satisfies some things.

By algebraic topology $MU_*(pt) \cong MU^*(pt)$ w/ reverse grading.

Nice Facts • $MU^*(X)$ has cup products (similar to H^*) so it's a graded algebra over MU^*.

• $MU^*(\mathbb{C}P^\infty) \cong MU^*[[x]]$ $\dim x = 2$

• $MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong MU^*[[x \otimes 1, 1 \otimes x]]$

• $\mathbb{C}P^\infty$ is an abelian topological group * up to homotopy

($K(\mathbb{Z},2)$. $B\mathbb{Z} = S^1$ is a $K(\mathbb{Z},1)$
BS^1 is a $K(\mathbb{Z},2)$ just like $\mathbb{C}P^\infty$)

we have multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{f} \mathbb{C}P^\infty$

$$MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xleftarrow{f^*} MU^*(\mathbb{C}P^\infty)$$

this is determined on $f^*(x) = F(x\otimes 1, 1\otimes x)$ which

is a formal group law so by Lazard's

$\exists! \theta : L \longrightarrow MU^*$.

Quillen's Theorem

The map $\theta : L \to MU^*$ is an isomorphism!

Remark

- Ignoring grading, we can think of $MU_*(X)$ as an L-module since $MU_* \cong MU^* \cong L$
- The formal group law assoc. to complex cobordism is universal.

More More More!

Def^n Let $\Gamma = \{x + b_1 x^2 + b_2 x^3 + \dots \in \mathbb{Z}[[x]]\}$ with composition $\circ$

$\Gamma \curvearrowright L$ like so:

For $\gamma \in \Gamma$ and $G(x,y)$ the universal formal group law, $\gamma^{-1} G(\gamma(x), \gamma(y))$ is another formal group law over L, so $\exists! \theta_\gamma : L \to L$ and since γ is invertible θ_γ is an automorphism

Definition

- Let $\mathcal{C}\Gamma$ denote the category of finitely presented L-modules with a Γ-action compatible with ours
- Let $\mathcal{FH}$ denote the category of finite CW complexes and homotopy classes of maps between them.

Remark We can regard $\overline{MU}^*$ as a functor
$$\mathcal{FH} \to \mathcal{C}G$$

Definition Let F be a formal group law.

let
$$[1](x) := x$$
$$[n](x) := F(x, [n-1](x))$$
and
$$[-n](x) = i([n](x))$$

these are the n-series of F.

They satisfy

$$[n](x) \equiv nx \mod x^2$$
$$[m+n](x) = F([m](x), [n](x))$$
$$[mn](x) = [m]([n](x))$$

Def^{n} Let $F(x,y)$ be a formal group law over a ring where the prime p is not a unit.
We say F has height h at p if the p-series

$$[p](x) \equiv a x^{p^h} + (\text{higher terms}) \bmod p.$$

with a invertible

$\boxed{\text{If } [p](x) \equiv 0 \bmod p \text{ then it has height } \infty \text{ at } p.}$

Ex

$$F(x,y) = x+y$$

Additive Formal Group Law

$F(x,y)$ has height ∞ since
$$[p](x) = px \equiv 0 \bmod p$$

$$F(x,y) = x+y+xy$$

Multiplicative Formal Group Law

$F(x,y)$ has height 1 since
$$[p](x) = (x+1)^p - 1 \equiv x^{p^1} \bmod p$$

Classification of Formal Group laws (Lazard)

Two formal group laws over the algebraic closure of $\mathbb{F}_p$ are

isomorphic $\Leftrightarrow$ they have the same height.

Let p be prime, and let's think about the universal formal group law $G(x,y)$

Let v_n be the coefficient of x^{p^n} in the series

$$[p](x) = G(x, [p-1](x))$$

Let $I_{p,n} \subset L$ be the ideal $(p, v_1, \dots v_{n-1})$

Invariant Prime Ideal Theorem (Morava, Landweber)

The only prime ideals in L that are invariant under Γ are $I_{p,n}$ where p is prime and $n \geq 0$

Moreover, in $L/I_{p,n}$ for $n > 0$, the subgroup

fixed by Γ is $\mathbb{Z}/p\mathbb{Z}[v_n]$

In L, the invariant subgroup is $\mathbb{Z}$.

$L = \mathbb{Z}[x_1, x_2, \dots]$ has crazy prime ideals, but these restrictions let us worry about far fewer in $\mathcal{C}\Gamma$.

Landweber Filtration Theorem Every module $M \in \mathcal{C}\Gamma$ admits a finite filtration by submodules in $\mathcal{C}\Gamma$

$$0 = F_0 M \subset F_1 M \subset \cdots \subset F_n M = M$$

where for each i, $F_i M / F_{i-1} M \cong L / I_{p,n}$ for some p and finite n.

Once we localize at , the only polynomial generators of are the $v_n =$ since others act freely.

That is , so if we tensor them away, we can study $V_p :=$ with a Γ-action instead.

is better for topology, formal group laws, computations, etc.

Falling Action

Corollary Let M be a p-local module and $x \in M$.

i) If $v_n^{-1} M = 0$, then

ii) If $x \neq 0$, then there is so that for all K.

i.e

M nontrivial $\Rightarrow$

iii) If $v_{n-1}^{-1} M = 0$, then so that multiplication by commutes with

iv) If $v_{n-1}^{-1} M \neq 0$, then so that commutes with

Definition • A p-local module in $\mathcal{C}\Gamma$ has if n is the smallest integer with

• A homomorphism $f : \Sigma^{d} M \longrightarrow M$ in $\mathcal{C}\Gamma$ is a if it induces:

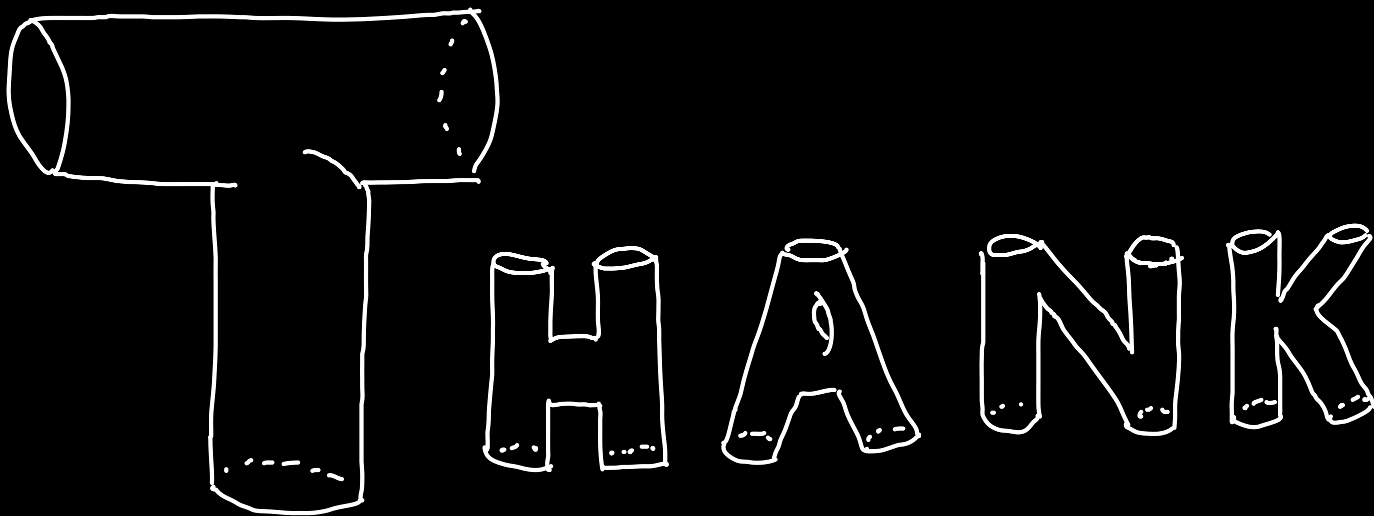
– an isomorphism in

– trivial homomorphism in

Corollary If M in $\mathcal{C}\Gamma$ is a p-local module with $v_n^{-1} M \neq 0$, then

3.4 will be covered by Cheng in 2 weeks.
Next week I'll talk about

4. Morava's Orbit Picture and Morava's Stabilizer Groups.



$\cancel{M}U^*$!